On the Linear Prediction of some L^p Random Fields*

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Abstract

This work is concerned with the prediction problem for a class of L^p–random fields. For this class of fields, we derive prediction error formulas, spectral factorizations, and orthogonal decompositions.

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1 Introduction

We study in these notes the prediction problem for a class of random fields which do not necessarily have finite variance. More precisely, we study zero mean random fields $\{X_{mn}\}$ for which there exist a finite nonnegative Borel measure μ on the torus and $p \in (1, \infty)$ with the property

$$E \left| \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_{mn} X_{mn} \right|^p \asymp \int \left| \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_{mn} e^{ims+int} \right|^p d\mu(e^{is}, e^{it}), \qquad (1)$$

for all $N, M = 0, 1, \ldots$, and all $a_{mn} \in \mathbf{C}$. In (1), $\asymp$ means that up to multiplicative constants, the two quantities are bounded above and below by each other. It is thus clear that the time domain of a random field satisfying (1) is isomorphic to $L^p(\mu)$, and prediction problems for $\{X_{mn}\}$ give rise to extremal problems in $L^p(\mu)$. Unless the equivalence in (1) becomes equality, metric projections are not preserved under the spectral isomorphism. Nevertheless, although we only give spectral domain results here, it is a simple matter to transfer these results to the time domain (see [7], for univariate results with $p = 2$). Random fields satisfying the condition (1) with equality and with $p = 2$ are well understood (they are the so–called homogeneous random fields). We seek to extend the prediction theory to the case p in the interval $(1, \infty)$, when the Hilbert space structure is replaced with the notion of Birkhoff–James orthogonality. This strategy has been carried out for one–parameter processes in [1, 2, 12]. The present work is concerned with the multiparameter case. We shall obtain prediction error formulas, spectral factorizations, and orthogonal (in some sense) decompositions of these L^p random fields. Examples of non-homogeneous fields satisfying (1) can also be obtained from the univariate results of [6].

2 Notation and Preliminaries

Let $\mathbf{T}$ be the unit circle in the complex plane, and let $d\sigma$ be normalized Lebesgue measure on $\mathbf{T}$. The torus $\mathbf{T}^2$ will be parameterized by the pair (e^{is}, e^{it}) throughout, and let $d\sigma_2 = d(\sigma \times \sigma)$ on $\mathbf{T}^2$.

Suppose that μ is a finite nonnegative Borel measure on $\mathbf{T}^2$. For any fixed parameter p, $1 < p < \infty$, the Banach space $L^p(\mu, \mathbf{T}^2)$ is reflexive and strongly convex, and is spanned by the set of functions $\{e^{ims+int} : (m,n) \in \mathbf{T}^2\}$. Every subset S of $\mathbf{T}^2$ determines a natural subspace of $L^p(\mu)$, namely that spanned by $\{e^{ims+int} : (m,n) \in S\}$. We write $\mathcal{M}(S,\mu)$ or $\mathcal{M}(S)$ to mean this subspace. For $S \subset \mathbf{Z}^2$, the notation $\Phi \hat{\in} S$, means that $\hat{\Phi} = 0$, outside of S.

Let x and y be elements of a Banach space $\mathcal{L}$. We write $x \perp_{\mathcal{L}} y$ if $\|x + \alpha y\|_{\mathcal{L}} \geq \|x\|_{\mathcal{L}}$ for all scalars α. Note that the relation $\perp_{\mathcal{L}}$ need not be symmetric. In the special case $\mathcal{L} = L^p(\mu)$, this notion of orthogonality has the following analytical characterization (See Singer [13], Theorem 1.11 and Lemma 1.14).

Lemma 2.1 *For f and g in $L^p(\mu)$, we have $f \perp_p g$ if and only if*

$$\int g\bar{f}|f|^{p-2}\, d\mu = 0,$$

where "0/0" is interpreted as zero.

It follows that the relation $\perp_p$ is linear in the second argument. We may thus write $f \perp_p \mathcal{M}$ for a subspace $\mathcal{M}$ with the obvious meaning. If it happens that two subspaces $\mathcal{M}$ and $\mathcal{N}$ have trivial intersection, and both $f \perp_p g$ and $g \perp_p f$ hold for all $f \in \mathcal{M}$ and $g \in \mathcal{N}$, then we write $\mathcal{M} \oplus_p \mathcal{N}$ for the algebraic sum of $\mathcal{M}$ and $\mathcal{N}$. This extends to finite sums in the obvious way.

3 Helson-Lowdenslager Halfplanes

The seminal paper on the prediction of homogeneous fields is one of Helson and Lowdenslager [5] (see also Wiener's collected works [11] for the history of the problem and further references). Many of their results carry over in a straightforward way from their framework to ours. Following their lead, we are concerned here with parameter sets $S \subset \mathbf{Z}^2$ which satisfy

1. $S \cup \{0\}$ is an additive semigroup;
2. $S \cup \{0\} \cup (\perp S) = \mathbf{Z}^2$;
3. $S \cap (\perp S) = \{\ \}$.

We write S_0 for $S \cup \{0\}$. Such a set is called a halfplane in the sense of Helson and Lowdenslager. Fix a halfplane S, and define

$$\epsilon^p = \inf \int |1+\phi|^p \, d\mu, \tag{2}$$

where the infimum is over $\phi \in \mathcal{M}(S)$. The infimum is achieved uniquely by some $\phi = H$. With that terminology, let us state the following two results:

Theorem 3.1 *Let $d\mu = w\, d\sigma_2 + d\eta$ be the Lebesgue decomposition of μ. If $\log w$ is integrable, then*

$$\epsilon^p = \exp \int \log w \, d\sigma_2;$$

otherwise $\epsilon = 0$.

Theorem 3.2 *Let S be a halfplane, and w a nonnegative summable function on $\mathbf{T}^2$. There exists $\Phi \in L^p(\sigma_2)$ such that $\Phi \hat{\in} S_0$ and $w = |\Phi|^p$, if and only if*

$$\int \log w \, d\sigma_2 > \perp\infty.$$

The first result is a generalization, to several variables, of Szego's infimum, the second is the corresponding outer factorization. Both proofs only require small adjustments from the corresponding proofs in [5].

4 Right Halfplanes

We now turn to the prediction problem associated with the parameter set R, given by

$$R = \{(m,n) : m \geq 1, \;\; n \in \mathbf{Z}\}. \tag{3}$$

Thus R is what we would call a right halfplane, a natural extension from the one-parameter case. For $p = 2$, prediction with respect to R was carried out in [8]. For general p, we shall obtain an error formula, and the corresponding spectral factorization. The methods of the previous section do not carry over directly: Unlike R, Helson-Lowdenslager halfplanes Λ enjoy certain algebraic properties which have deep analytical consequences. Here, some elementary arguments are needed to reduce the problem to the one-parameter case.

This is the error formula.

Theorem 4.1 *Let R be the halfplane defined in (3), and let $d\mu$ be a finite nonnegative Borel measure on $\mathbf{T}^2$. Write*

$$d\mu = w_R\, d(\sigma \times \mu_2) \ + \ d\lambda_R,$$

where μ_2 is the second marginal measure of μ and $\lambda_R \perp (\sigma \times \mu_2)$. Then

$$\inf\{\int |1+\phi|^p\, d\mu : \quad \phi \hat{\in} R\} = \int [\exp \int \log w_R(e^{is}, e^{it})\, d\sigma(e^{is})]\, d\mu_2(e^{it}),$$

where the right side is interpreted as zero if the integral diverges.

Proof Let G be the unique member of $\mathcal{M}(R, \mu)$ for which

$$\begin{aligned} \delta^p &= \inf\{\int |1+\phi|^p\, d\mu : \phi \hat{\in} R\} \qquad (4) \\ &= \int |1+G|^p\, d\mu. \end{aligned}$$

Assume for now that $\delta > 0$. The above extremal condition implies that

$$(1+G) \perp_p (1+G) e^{ims+int}$$

for all $(m, n) \in R$. By Lemma 2.1, this gives

$$\begin{aligned} 0 &= \int |1+G|^{p-2} (1+\bar{G})(1+G) e^{ims+int}\, d\mu \\ &= \int |1+G|^p e^{ims+int}\, d\mu \qquad (5) \end{aligned}$$

for all $(m, n) \in R$. Taking complex conjugates, we see that this holds whenever $m \neq 0$. Let $d\nu = |1+G|^p\, d\mu$, and let ν_2 be the second marginal measure of ν. Then for any half open arc V of $\mathbf{T}$ the indicator function χ_V can be estimated boundedly pointwise by trigonometric polynomials. Thus equation (5) gives

$$\int e^{ims} \chi_V(e^{it})\, d\nu = 0$$

whenever $m \neq 0$. This remains true with V replaced by any Borel subset of $\mathbf{T}$. Likewise, for any half open arc U of $\mathbf{T}$, the indicator χ_U can be estimated boundedly pointwise by trigonometric polynomials of the form $f(e^{is}) = \sum a_j e^{ijs}$. For such polynomials, we have

$$\begin{aligned} \int f(e^{is}) \chi_V(e^{it})\, d\nu &= \sum a_m \int e^{ims} \chi_V(e^{it})\, d\nu \\ &= \int f(e^{is})\, d\sigma(e^{is}) \cdot \nu_2(V). \end{aligned}$$

It follows that

$$\begin{aligned} \nu(U \times V) &= \int \chi_{U \times V}\, d\nu \\ &= \int \chi_U\, d\sigma \cdot \nu_2(V) \\ &= (\sigma \times \nu_2)(U \times V). \end{aligned}$$

This remains true if the rectangle $U \times V$ is replaced by any Borel subset of $\mathbf{T}^2$, and consequently, $\nu = \sigma \times \nu_2$. Let μ_2 be the second marginal measure of μ. Note that $\mu_2(V) = 0$ implies that $\mu(\mathbf{T} \times V) = 0$, and in turn this requires

$$\begin{aligned} 0 &= \int_{\mathbf{T} \times V} |1 + G|^p\, d\mu \\ &= \nu(\mathbf{T} \times V) \\ &= \nu_2(V). \end{aligned}$$

Hence $\nu_2 << \mu_2$, and we may write $d\nu_2 = q\, d\mu_2$ for some density function q.

Let μ have the Lebesgue decomposition

$$d\mu = w_R\, d(\sigma \times \mu_2) + d\lambda_R.$$

By the above observations,

$$\begin{aligned} |1 + G|^p w_R\, d(\sigma \times \mu_2) + |1 + G|^p\, d\lambda_R &= |1 + G|^p\, d\mu \\ &= d\nu \\ &= d(\sigma \times \nu_2) \\ &= q d(\sigma \times \mu_2). \end{aligned}$$

It follows that $d\lambda_R$ annihilates $|1 + G|^p$, and $q = |1 + G|^p u$, a.e.$[\sigma \times \mu_2]$.

The orthogonality condition $(1 + G) \perp_p e^{ims+int}$, $m \geq 1$, provides

$$\int |1 + G|^{p-2}(1 + \bar{G}) e^{ims+int}\, d\mu,$$

$m \geq 1$. We may replace $d\mu$ with $w_R\, d(\sigma \times \mu_2)$. With that, the above can be reinterpreted as $(1 + G) \perp_p e^{ims+int}$ in the geometry of $L^p(w_R\, d(\sigma \times \mu_2))$.

Let

$$A = \{e^{it} : \int \log w_R(e^{is}, e^{it})\, d\sigma(e^{is}) > \perp\infty\}. \tag{6}$$

Define the function $\Psi(e^{is}, e^{it})$ by

$$\Psi(e^{is}, e^{it}) = \lim_{r \to 1-} \exp \frac{1}{p} \int \frac{e^{i\theta} + re^{is}}{e^{i\theta} \perp re^{is}} \log w_R(e^{i\theta}, e^{it})\, d\sigma(e^{i\theta}) \tag{7}$$

for $e^{it} \in A$, otherwise zero. Then $\Psi(\cdot, e^{it})$ is outer for μ_2-almost every e^{it} in A, and

$$|\Psi|^p = w_R \chi_{\mathbf{T} \times A}.$$

Suppose for the present that $\mu_2(A) = \mu_2(\mathbf{T})$. Then, with the help of the Lemma below, we have

$$\begin{aligned}
\delta^p &= \int |1 + G|^p \, d\mu \\
&= \int |1 + G|^p w_R \chi_A \, d(\sigma \times \mu_2) \\
&= \inf\{\int |1 + Q|^p |\Psi|^p \, d(\sigma \times \mu_2) : Q \hat{\in} R\} \\
&= \inf\{\int |\Psi + Q\Psi|^p \, d(\sigma \times \mu_2) : Q \hat{\in} R\} \\
&= \inf\{\int |\Psi + Q|^p \, d(\sigma \times \mu_2) : Q \hat{\in} R\} \\
&= \int |\Psi(0, e^{it})|^p \, d\mu_2(e^{it}) \\
&= \int [\exp \int \log w_R(e^{is}, e^{it}) \, d\sigma(e^{is})] \, d\mu_2(e^{it}).
\end{aligned}$$

On the other hand, suppose that $\mu_2(A) = 0$. We may apply the previous calculation to the measure

$$d\mu + \frac{1}{n} d(\sigma \times \mu_2)$$

to get

$$\begin{aligned}
\delta^p &\leq \inf\{\int |1 + Q|^p d\mu + \frac{1}{n} d(\sigma \times \mu_2) : \quad Q \hat{\in} R\} \\
&= \int [\exp \int \log[w_R(e^{is}, e^{it}) + \frac{1}{n}] \, d\sigma(e^{is})] \, d\mu_2(e^{it}).
\end{aligned}$$

As n increases without bound, the last expression approaches zero by monotone convergence. Hence $\delta = 0$ in this case.

In general, $\mu_2(A)$ falls between these two extremes. But $\chi_A(e^{it})$ is estimable boundedly pointwise by trigonometric polynomials, and so for any $Q\hat{\in}R$ we have

$$1+Q=(1+Q)\chi_A+(1+Q)\chi_{A^c};$$

the point is that both $Q\chi_A$ and $Q\chi_{A^c}$ lie in $\mathcal{M}(R)$. Thus

$$\begin{aligned}\delta^p &= \inf\{\int|1+Q|^p\,d\mu: \quad Q\hat{\in}R\} \\ &= \inf\{\int|1+Q_1|^p\chi_A\,d\mu: \quad Q_1\hat{\in}R\} \\ &\quad +\inf\{\int|1+Q_2|^p\chi_{A^c}\,d\mu: \quad Q_2\hat{\in}R\}.\end{aligned}$$

If we assume that $\delta=0$, then we may still construct the function Ψ as before, and then

$$\begin{aligned}0 &= \delta^p \\ &\geq \inf\int|1+Q|^p w_R\chi_A\,d(\sigma\times\mu_2) \\ &= \int_A[\exp\int\log w_R(e^{is},e^{it})\,d\sigma(e^{is})]\,d\mu_2(e^{it}) \\ &\geq 0.\end{aligned}$$

Equality is forced throughout, which gives $\mu_2(A)=0$, and once again

$$\delta^p=\int[\exp\int\log w_R(e^{is},e^{it})\,d\sigma(e^{is})]\,d\mu_2(e^{it})$$

This completes the proof. □

Part of the argument in the proof above relied on this next lemma, which asserts that the closed linear span of the random field and of its innovation are identical.

Lemma 4.2 *Let $w_R(e^{is},e^{it})$ be nonnegative and integrable with respect to $\sigma\times\mu_2$, and assume that $\log w_R$ is $\sigma(e^{is})$-integrable almost everywhere-$\mu_2(e^{it})$. Then with Ψ as defined in (7)*

$$\begin{aligned}&L^p(\chi_A(e^{it})\,d(\sigma\times\mu_2))\text{-span}\{e^{ims+int}\Psi(e^{is},e^{it}): \quad (m,n)\in R\} \\ = \;&L^p(\chi_A(e^{it})\,d(\sigma\times\mu_2))\text{-span}\{e^{ims+int}: \quad (m,n)\in R\}.\end{aligned}$$

Proof The right side is $\mathcal{M}(R, \chi_A\, d(\sigma \times \mu_2))$, which obviously contains the left side as a subspace. Let l_0 be a bounded linear functional on

$$\mathcal{M}(R, \chi_A\, d(\sigma \times \mu_2))$$

which annihilates every $\Psi(e^{is}, e^{it})e^{ims+int}$ for all $(m,n) \in R$. Hahn-Banach gives a norm-preserving extension l of l_0 to all of $L^p(\chi_A\, d(\sigma \times \mu_2))$. There is a function $h(e^{is}, e^{it})$ in the dual space $L^{p'}(\chi_A\, d(\sigma \times \mu_2))$ such that

$$l(f) = \int fh(\chi_A\, d(\sigma \times \mu_2))$$

for all f in $L^p(\chi_A\, d(\sigma \times \mu_2))$. In particular,

$$0 = \int e^{ims+int}\Psi h\chi_A\, d(\sigma \times \mu_2)$$

for all $(m,n) \in R$. Hence

$$\int e^{ims}\Psi h\chi_A\, d\sigma(e^{is})$$

must vanish almost everywhere-μ_2. For such e^{it} the function $\Psi(\cdot, e^{it})h(\cdot, e^{it})$ lies in the Hardy class $H^1(\mathbf{T})$. Since $\Psi(\cdot, e^{it})$ is outer for $e^{it} \in A$, we have that $h(\cdot, e^{it})$ is of Nevanlinna class. Consequently, h must annihilate all of $\mathcal{M}(R, \chi_A\, d(\sigma \times \mu_2))$. This establishes the reverse inequality, and hence the claim. □

Note that with the definitions of this section, the measure μ has the decomposition

$$d\mu = |\Psi|^p\, d(\sigma \times \mu_2) + w_R\chi_{A^c}\, d(\sigma \times \mu_2) + d\lambda_R.$$

Evidently, the innovation space corresponds to the first component of this decomposition.

Theorem 4.3 *We have*

$$\begin{aligned} & L^p(\mu)\text{-span}\{(1+G)e^{ims+int} : \ (m,n) \in R\} \\ = \ & L^p(\chi_A w_R\, d(\sigma \times \mu_2))\text{-span}\{(1+G)e^{ims+int} : \ (m,n) \in R\} \\ = \ & L^p(\chi_A w_R\, d(\sigma \times \mu_2))\text{-span}\{e^{ims+int} : \ (m,n) \in R\}. \end{aligned}$$

Proof The first equality holds since $1+G$ is annihilated by $w_R \chi_{A^c}\, d(\sigma \times \mu_2)$ $+\, d\lambda_R$. The second follows from observing that $w_R = q/|1+G|^p$, and so for all f and $Q \,\hat{\in} R$,

$$\begin{aligned} & \int |f + (1+G)Q|^p \chi_A w_R \, d(\sigma \times \mu_2) \\ = & \int |f + (1+G)Q|^p \chi_A \frac{q}{|1+G|^p} \, d(\sigma \times \mu_2) \\ = & \int |f\Psi + q^{1/p} Q|^p \chi_A \, d(\sigma \times \mu_2). \end{aligned}$$

In the last inequality, $f\Psi$ lies in $\mathcal{M}(R, \chi_A \, d(\sigma \times \mu_2))$; furthermore, $q^{1/p}$ is a function of e^{it} only, and hence $q^{1/p}Q$ remains in $\mathcal{M}(R, \chi_A \, d(\sigma \times \mu_2))$. The last expression can be made arbitrarily small by choosing $Q \hat{\in} R$ appropriately. This shows that any $f \hat{\in} R$ can be approximated by $(1+G)Q$, where Q belongs to $\mathcal{M}(R, \chi_A \, d(\sigma \times \mu_2))$. The claim follows. $\square$

5 Outer Properties

Define Λ to be the parameter set

$$\Lambda = \{(m,n) \in \mathbf{T}^2 : \quad m = 0, n \geq 1\} \cup \{(m,n) \in \mathbf{T}^2 : \quad m \geq 1\},$$

and as usual put $\Lambda_0 = \Lambda \cup \{0\}$. Thus Λ is a halfplane in the sense of Helson and Lowdenslager. Whereas geometric arguments are used in [5] to obtain a spectral factorization, here the special case Λ yields an explicit analytical formula. The method is adapted from that used in [9] for the case $p = 2$, [10] for matrix valued functions, and [3] for operator valued functions. The resulting outer factor is used in later sections to derive orthogonal decompositions of the field.

Assume that the prediction error ϵ from (3) is positive, so that the density function w has integrable logarithm. We construct the outer function Φ from Theorem 3.2 analytically as follows. Define

$$\begin{aligned} \beta(z, e^{it}) &= \exp \frac{1}{p} \int \frac{e^{i\theta} + z}{e^{i\theta} \perp z} \log w(e^{is}, e^{it}) \, d\sigma(e^{i\theta}) \\ \alpha(z) &= \exp \int \frac{e^{i\theta} + z}{e^{i\theta} \perp z} \log \beta(0, e^{i\theta}) \, d\sigma(e^{i\theta}) \\ \Phi(e^{is}, e^{it}) &= \frac{\alpha(e^{it})\beta(e^{is}, e^{it})}{\beta(0, e^{it})}, \end{aligned}$$

with radial limits taken in accordance with Fatou's Theorem. Check that the following conditions hold:

$$\begin{array}{rcl} |\Phi|^p & = & w \\ |\beta|^p & = & w \\ \Phi & \hat{\in} & \Lambda \\ \beta & \hat{\in} & R. \end{array}$$

Furthermore, $\Phi(\cdot, e^{it})$ is outer a.e.$[\sigma(e^{it})]$, and $\Phi(0, \cdot)$ is outer.

And now we find that Φ has an outer property with respect to Λ.

Proposition 5.1 *With the above definitions we have*

$$\begin{array}{rl} & L^p(\sigma_2)\text{-span}\{e^{ims+int}\Phi : (m,n) \in \Lambda\} \\ = & L^p(\sigma_2)\text{-span}\{e^{ims+int} : (m,n) \in \Lambda\} \end{array}$$

Proof By Lemma 4.2 the left side contains the $L^p(\sigma_2)$-span of $\{e^{ims+int} : m \geq 1\}$. Hence it also contains $e^{int}\Phi(0, e^{it})$ for each $n \geq 1$, since

$$e^{int}\Phi(0, e^{it}) = e^{int}\Phi(e^{is}, e^{it}) \perp e^{int}[\Phi(e^{is}, e^{it}) \perp \Phi(0, e^{it})].$$

Now Beurling provides that the left side must therefore contain every e^{int} for each $n \geq 1$. Thus the inclusion $\supseteq$ holds. The reverse inclusion is obviously true. □

The following states that the innovation part of the field is associated with the continuous part of the measure.

Theorem 5.2

$$\begin{array}{rl} & L^p(\mu)\text{-span}\{(1+H)e^{ims+int} : (m,n) \in \Lambda\} \\ = & L^p(w\, d\sigma_2)\text{-span}\{(1+H)e^{ims+int} : (m,n) \in \Lambda\} \\ = & L^p(w\, d\sigma_2)\text{-span}\{e^{ims+int} : (m,n) \in \Lambda\}. \end{array}$$

Proof The first equation holds since $1 + H$ was shown to vanish on the singular part of μ. For the second equality, let $f \hat{\in} \Lambda$ and $Q \hat{\in} \Lambda$ and note that

$$\int |f + (1+H)Q|^p w\, d(\sigma_2) = \int |\Phi f + \epsilon Q|^p\, d(\sigma_2).$$

For f fixed, this can be made arbitrarily small by choice of Q. Thus the inclusion $\supseteq$ holds in the second relation. The reverse inclusion is obvious. □

6 Three Part Decomposition

In [5], a second-order stationary random field is decomposed orthogonally into its regular, evanescent, and singular parts. In this section the corresponding result for L^p is established. In this setting, Hilbert space arguments give way to more elementary methods, and the usual notion of orthogonality is replaced by Birkhoff-James orthogonality. We find that, perhaps surprisingly, the component spaces of the decomposition are related by $\oplus_p$, even though the orthogonality $\perp_p$ is generally not symmetric.

We form the Lebesgue decompositions of μ and its second marginal, μ_2.

$$\begin{aligned} d\mu &= w_R\, d(\sigma \times \mu_2) + d\lambda_R \\ d\mu_2 &= u_2\, d\sigma + d\eta_2. \end{aligned}$$

There is a Borel set Ξ of $\mathbf{T}$ such that $\eta_2(\Xi) = \sigma(\Xi^c) = 0$. Put $w = w_R u_2$, and define A as in equation (6). Now put

$$\begin{aligned} d\mu_r &= \begin{cases} w\, d(\sigma_2), & \log w \in L^1(\sigma_2) \\ 0, & \log w \notin L^1(\sigma_2). \end{cases} \\ d\mu_e &= \begin{cases} \chi_{A\times\Xi^c} w_R\, d(\sigma \times \eta_2), & \log w \in L^1(\sigma_2) \\ \chi_A w_R\, d(\sigma \times \mu_2), & \log w \notin L^1(\sigma_2). \end{cases} \\ d\mu_s &= \chi_{A^c}\, d(\sigma \times \mu_2) + d\lambda_R. \end{aligned}$$

Then $d\mu = d\mu_r + d\mu_e + d\mu_s$, a pairwise singular decomposition.

Define the subspace $\mathcal{R}$ of $L^p(\mu)$ as follows.

$$\mathcal{R} = L^p(\mu)\text{-span}\{e^{ims+int}(1 + H) : (m, n) \in \mathbf{Z}^2\}.$$

By Theorem 5.2, we can identify $\mathcal{R}$ with $L^p(\mu_r)$.

Furthermore define $\mathcal{S}$ to be the subspace $\bigcap_{m=0}^{\infty} \mathcal{M}(\Lambda, \mu)$ of $L^p(\mu)$. Note that $\mathcal{S} = \bigcap_{m=0}^{\infty} \mathcal{M}(R, \mu)$ as well.

If $f \in \mathcal{S}$, then $e^{ims+int} f \in e^{is}\mathcal{M}(\pi, \mu)$ for all $(m, n) \in \mathbf{Z}^2$. Hence $(1 + G)\perp_p e^{ims+int} f$ for all (m, n). By Lemma 2.1 this gives

$$0 = \int |1 + G|^{p-2}(1 + G)e^{ims+int} f\, d\mu$$

for all (m, n). It follows that $(1 + G)f = 0$. By Theorem 4.3, f vanishes a.e.$[\chi_A w_R\, d(\sigma \times \mu_2)]$. It follows that $f \in L^p(\mu_s)$, and we get $\mathcal{S} \subseteq L^p(\mu_s)$.

On the other hand, let Θ be a Borel set such that $\mu_S(\Theta^c) = 0$ and $(\mu \perp \mu_S)(\Theta) = 0$. Then the infimum

$$\gamma^p = \inf\{\int |\chi_\Theta \perp \phi|^p \, d\mu : \quad \phi \hat{\in} R\}$$

is attained by some $\phi = G_\Theta$ belonging to $\mathcal{M}(R, \mu)$. Arguing as before, we get that $(\chi_\Theta \perp G_\Theta) \, d\lambda_R = 0$. Thus,

$$\begin{aligned}
\gamma^p &= \int |\chi_\Theta \perp G_\Theta|^p \, d\mu \\
&= \int |\chi_\Theta \perp G_\Theta|^p u \, d(\sigma \times \mu_2) \\
&= \inf \int |\chi_\Theta \perp \phi|^p u \, d(\sigma \times \mu_2) \\
&= \inf(\int |\chi_\Theta \perp \phi_1|^p \chi_A u \, d(\sigma \times \mu_2) \\
&\quad + \int |\chi_\Theta \perp \phi_2|^p \chi_{A^c} u \, d(\sigma \times \mu_2)) \\
&= \inf \int |\chi_\Theta \perp \phi_1|^p \chi_A u \, d(\sigma \times \mu_2) \\
&\quad + \inf \int |\chi_\Theta \perp \phi_2|^p \chi_{A^c} u \, d(\sigma \times \mu_2)
\end{aligned}$$

Indeed, equality holds in the last step (rather than $\geq$) since the two terms of

$$\phi = \chi_{`}\phi + \chi_{A^c}\phi$$

both belong to $\mathcal{M}(R, \mu)$.

It follows that χ_Θ belongs to $\mathcal{M}(R, \mu)$. Similarly, we find that $\chi_\Theta e^{ims+int} \in \mathcal{M}(R, \mu)$ for all (m, n). This gives $L^p(\mu_s) \subseteq \mathcal{S}$, and hence the subspaces are equal.

We summarize and extend these results below.

Theorem 6.1

(i)

$$d\mu = d\mu_r + d\mu_e + d\mu_s.$$

(ii)

$$L^p(\mu) = \mathcal{R} \oplus_p \mathcal{E} \oplus_p \mathcal{S}.$$

(iii)

$$\begin{aligned}\mathcal{R} &= L^p(d\mu_r)\\ \mathcal{E} &= L^p(d\mu_e)\\ \mathcal{S} &= L^p(d\mu_s).\end{aligned}$$

(iv)

$$\mathcal{M}(\Lambda,\mu) = \mathcal{M}(\Lambda,\mu_r) \oplus_p \mathcal{M}(\Lambda,\mu_e) \oplus_p \mathcal{M}(\Lambda,\mu_s).$$

(v)

$$\begin{array}{lll} (\mu_r)_r = \mu_r & (\mu_e)_r = 0 & (\mu_s)_r = 0 \\ (\mu_r)_e = 0 & (\mu_e)_e = \mu_e & (\mu_s)_e = 0 \\ (\mu_r)_s = 0 & (\mu_e)_s = 0 & (\mu_s)_s = \mu_s. \end{array}$$

Proof Statements *(i)*, *(ii)* and *(iii)* were already established previously. Assertion *(iv)* follows from Theorem [5] and the fact that $\mathcal{M}(\Lambda,\mu_S) = \mathcal{S}$. Claim *(v)* is straightforward to verify from the definitions. □

Thus, we see that the field does indeed decompose into its regular, evanescent, and singular parts; furthermore, the measure μ decomposes in a corresponding way. The decomposition respects subspaces generated by Λ: In particular, it is significant that the component spaces in *(iv)* are already subspaces of $\mathcal{M}(\Lambda,\mu)$. The condition *(v)* is a sort of inertial property: It asserts that each of the subspaces $\mathcal{R}$, $\mathcal{E}$ and $\mathcal{S}$ itself decomposes in a trivial way under this scheme.

7 Four Part Decomposition

We now consider a decomposition of $L^p(\mu)$ with respect to both vertical and horizontal notions of regularity. The decomposition will consist of four components, one which is regular with respect to both the vertical and horizontal shifts, one which is remote in both shifts, and two which represent the mixed types. For the $p = 2$ case such decompositions have been established in [4, 8, 9]. For general p, we find that the component spaces are themselves L^p spaces for some measure, and they are related by the symmetric orthogonal sum $\oplus_p$.

To begin, define

$$
\begin{aligned}
R_0 &= \{(m,n) \in \mathbf{T}^2 : \quad m \geq 0\} \\
T_0 &= \{(m,n) \in \mathbf{T}^2 : \quad n \geq 0\} \\
T &= \{(m,n) \in \mathbf{T}^2 : \quad n > 0\} \\
\mathcal{S}_R &= \mathcal{S} \\
&= \cap_{m=0}^{\infty} e^{ims}\mathcal{M}(R) \\
\mathcal{S}_T &= \cap_{n=0}^{\infty} e^{int}\mathcal{M}(T).
\end{aligned}
$$

Thus R_0 is a shift of the right halfplane R previously used, and T_0 is its counterpart along the orthogonal direction; $\mathcal{S}_R$ is the remote space written before as simply $\mathcal{S}$, while $\mathcal{S}_T$ is its rotated counterpart.

Next, form the Lebesgue decompositions

$$
\begin{aligned}
d\mu &= w_R\, d(\sigma \times \mu_2) + d\lambda_R \\
d\mu &= w_T\, d(\mu_1 \times \sigma) + d\lambda_T.
\end{aligned}
$$

There exist measurable subsets Γ and Δ of $\mathbf{T}^2$ such that

$$
\begin{aligned}
\lambda_R(\Gamma) &= 0 \\
(\sigma \times \mu_2)(\Gamma^c) &= 0 \\
\lambda_T(\Delta) &= 0 \\
(\mu_1 \times \sigma)(\Delta^c) &= 0
\end{aligned}
$$

and furthermore we define subsets A and B of $\mathbf{T}$ by

$$
\begin{aligned}
A &= \{e^{it} : \quad \int \log w_R(e^{is}, e^{it})\, d\sigma(e^{is}) > \perp\infty\} \\
B &= \{e^{is} : \quad \int \log w_T(e^{is}, e^{it})\, d\sigma(e^{it}) > \perp\infty\}.
\end{aligned}
$$

Let $\mathcal{K}$ be the span of the functions $e^{ims+int}(1+G)$, where G is the extremal function from (4).

We show that $\mathcal{S}_R$ has an orthogonal complement in $L^p(\mu)$.

Proposition 7.1

$$L^p(\mu) = \mathcal{K} \oplus_p \mathcal{S}_R.$$

Proof Write $\mathcal{I} = \vee_{n=-\infty}^{\infty} e^{int}(1+G)$. We know that

$$\mathcal{M}(R_0) = \mathcal{I} + e^{is}\mathcal{M}(R_0),$$

and consequently

$$\mathcal{M}(R_0) = \mathcal{K}_N + e^{i(N+1)s}\mathcal{M}(R_0),$$

where

$$\mathcal{K}_N = \mathcal{I} + e^{is}\mathcal{I} + e^{2is}\mathcal{I} + \cdots + e^{iNs}\mathcal{I}.$$

Let $f \in \mathcal{M}(R_0)$, and write $f = k_n + m_n$, where $k_n \in \mathcal{K}_n$ and $m_n \in e^{i(n+1)s}\mathcal{M}(R_0)$. Since

$$\mathcal{K}_n \perp_p e^{i(n+1)s}\mathcal{M}(R_0)$$

for all n, the decomposition is unique. Observe that

$$\|k_n\| \leq \|k_n + m_n\| = \|f\|$$

and

$$\|m_n\| \leq \|f \perp k_n\| \leq 2\|f\|.$$

Thus we can find weakly convergent subsequences of $\{k_n\}$ and $\{m_n\}$ with limits k_∞ and m_∞, respectively. Then

$$\begin{aligned} k_\infty &\in \vee_{n=0}^{\infty}\mathcal{K}_n \\ m_\infty &\in \mathcal{S}_R \\ f &= k_\infty + m_\infty. \end{aligned}$$

This shows that

$$\mathcal{M}(R) = \vee_{n=0}^{\infty}\mathcal{K}_n + \mathcal{S}_R.$$

It follows

$$\begin{aligned} L^p(\mu) &= \vee_{m=-\infty}^{0} e^{ims}\mathcal{M}(R) \\ &= \vee_{m=-\infty}^{\infty} e^{ims}\mathcal{I}_n + \mathcal{S}_R \\ &= \mathcal{K} + \mathcal{S}_R. \end{aligned} \tag{8}$$

From Theorem 4.3 we have

$$\mathcal{K} = L^p(\chi_A w_R \, d(\sigma \times \mu_2)).$$

Observe that if $g \in \mathcal{S}_R$, then

$$e^{ims+int}(1+G) \perp_p g$$

for all (m,n). Thus

$$|1+G|^{p-2}(1+\bar{G})ge^{-ims-int}\chi_A w_R = 0,$$

almost everywhere-$[\sigma \times \mu_2]$. But $(1+G)$ is essentially nonvanishing on $\Gamma \cap (\mathbf{T} \times A)$, forcing $g\chi_A = 0$ almost everywhere-$(\sigma \times \mu_2)$. This proves that g lies in $L^p(\chi_{A^c} w_R \, d(\sigma \times \mu_2) + d\lambda)$, and hence as does all of $\mathcal{S}_R$. In fact

$$\begin{aligned} \mathcal{S}_R &= L^p(\chi_{(\mathbf{T}\times A^c)\cup\Gamma^c} \, d\mu) \\ &= L^p(\chi_{A^c}) w_R \, d(\sigma \times \mu_2) + d\lambda). \end{aligned}$$

Finally, since $\chi_{(\mathbf{T}\times A^c)\cup\Gamma^c}$ and $\chi_{(\mathbf{T}\times A)\cap\Gamma}$ indicate disjoint sets, we may use $\oplus_p$ in place of $+$ in (8). $\square$

Let us write

$$\begin{aligned} E &= (\mathbf{T} \times A) \cap \Gamma \\ F &= (B \times \mathbf{T}) \cap \Delta. \end{aligned}$$

The following was also proved above. The point is that the remote spaces $\mathcal{S}_R$ and $\mathcal{S}_T$ are themselves L^p spaces for some measure.

Corollary 7.2

$$\begin{aligned} \mathcal{S}_R &= L^p(\chi_{E^c} \, d\mu) \\ \mathcal{S}_T &= L^p(\chi_{F^c} \, d\mu) \end{aligned}$$

Accordingly, the complements of $\mathcal{S}_R$ and $\mathcal{S}_T$ are L^p spaces, and we can naturally associate $L^p(\mu)$ with the four part decomposition

$$L^p(\mu) = \mathcal{L}_a \oplus_p \mathcal{L}_b \oplus_p \mathcal{L}_c \oplus_p \mathcal{L}_d, \tag{9}$$

where

$$\begin{aligned} \mathcal{L}_a &= L^p(\chi_{E\cap F} \, d\mu) \\ \mathcal{L}_b &= L^p(\chi_{E\cap F^c} \, d\mu) \\ \mathcal{L}_c &= L^p(\chi_{E^c\cap F} \, d\mu) \\ \mathcal{L}_d &= L^p(\chi_{E^c\cap F^c} \, d\mu). \end{aligned}$$

Thus $\mathcal{L}_a$ is the part of $L^p(\mu)$ which is both horizontally regular and vertically regular; $\mathcal{L}_d$ is both horizontally singular and vertically singular; $\mathcal{L}_b$ and $\mathcal{L}_c$ are of the mixed types.

8 An Inertial Property

With the four part decomposition (9) established, we examine the behavior of its component spaces. Since $\mathcal{L}_a$ is itself an L^p space, it has a decomposition analogous to (9):

$$\mathcal{L}_a = \mathcal{L}_{aa} \oplus_p \mathcal{L}_{ab} \oplus_p \mathcal{L}_{ac} \oplus_p \mathcal{L}_{ad},$$

and similarly with $\mathcal{L}_b$, $\mathcal{L}_c$ and $\mathcal{L}_d$. It is desirable for their component spaces to have the following inertial property.

$$\left.\begin{array}{llll} \mathcal{L}_{aa} = \mathcal{L}_a & \mathcal{L}_{ab} = 0 & \mathcal{L}_{ac} = 0 & \mathcal{L}_{ad} = 0 \\ \mathcal{L}_{ba} = 0 & \mathcal{L}_{bb} = \mathcal{L}_b & \mathcal{L}_{bc} = 0 & \mathcal{L}_{bd} = 0 \\ \mathcal{L}_{ca} = 0 & \mathcal{L}_{cb} = 0 & \mathcal{L}_{cc} = \mathcal{L}_c & \mathcal{L}_{cd} = 0 \\ \mathcal{L}_{da} = 0 & \mathcal{L}_{db} = 0 & \mathcal{L}_{dc} = 0 & \mathcal{L}_{dd} = \mathcal{L}_d. \end{array}\right\} \tag{10}$$

This would say that the component spaces themselves decompose trivially under (9). Theorem 6.1*(v)* provides that the three part decomposition has this inertial property; it turns out, however, that the four part decomposition does not. Rather, the following theorem shows that (10) is equivalent to three separate criteria developed below: condition (11), a constraint on the underlying measure μ; (12), a property of the halfplane subspaces $\mathcal{M}(R_0)$ and $\mathcal{M}(T_0)$; and (13) a condition on the associated metric projection operators. For $p = 2$, a part of this result was previously known [9, Theorem III.12].

Theorem 8.1 *The four conditions (10), (12), (13) and (11) are equivalent.*

Proof To begin, assume that (10) holds, and further decompose μ as follows

$$\begin{aligned} d\mu_1(e^{is}) &= u_1(e^{is})\, d\sigma(e^{is}) + d\eta_1(e^{is}) \\ d\mu_2(e^{it}) &= u_2(e^{it})\, d\sigma(e^{it}) + d\eta_2(e^{it}) \\ d\lambda_R &= v_R\, d(\lambda_{R,1} \times \sigma) + d\xi_R \\ d\lambda_T &= v_T\, d(\sigma \times \lambda_{T,2}) + d\xi_T, \end{aligned}$$

where $\lambda_{(\cdot),j}$ is the jth marginal measure of $\lambda_{(\cdot)}$. We substitute these into the definitions of $\chi_a\,d\mu$, $\chi_b\,d\mu$, $\chi_c\,d\mu$, and $\chi_d\,d\mu$, and impose consistency. This forces the following identifications:

$$
\begin{aligned}
\chi_a w_R u_2\,d\sigma_2 &= \chi_a w_T u_1\,d\sigma_2 \\
\chi_a w_R\,d(\sigma\times\eta_2) &= 0 \\
\chi_a w_T\,d(\eta_1\times\sigma) &= 0 \\
\chi_b w_R u_2\,d\sigma_2 &= \chi_b w_T u_1\,d\sigma_2 \\
\chi_b w_R\,d(\sigma\times\eta_2) &= \chi_b w_T v_2\,d(\sigma\times\lambda_{T,2}) \\
\chi_c w_R u_2\,d\sigma_2 &= \chi_c w_T u_1\,d\sigma_2 \\
\chi_c w_T\,d(\eta_1\times\sigma) &= \chi_c w_R v_R\,d(\lambda_{R,1}\times\sigma) \\
\chi_b\,d\xi_T &= 0 \\
\chi_c\,d\xi_R &= 0 \\
\chi_d\,d\xi_R &= \chi_d\,d\xi_T \\
\chi_d w_R u_2\,d\sigma_2 &= \chi_d w_T u_1\,d\sigma_2 \\
\chi_d w_R\,d(\sigma\times\eta_2) &= \chi_d w_T v_T\,d(\sigma\times\lambda_{T,2}) \\
\chi_d w_T\,d(\eta_1\times\sigma) &= \chi_d w_R v_R\,d(\lambda_{R,1}\times\sigma).
\end{aligned}
$$

Put $\chi_d\,d\xi_R$ $(=\chi_d\,d\xi_T) = d\lambda$ and $w = w_R u_2$ $(= w_T u_1)$. With these and the existing definitions, this in turn gives

$$
\begin{aligned}
\chi_a\,d\mu &= \chi_a w\,d\sigma_2 \\
\chi_b\,d\mu &= \chi_b w\,d\sigma_2 + \chi_b w_R\,d(\sigma\times\eta_2) \\
\chi_c\,d\mu &= \chi_c w\,d\sigma_2 + \chi_c w_T\,d(\eta_1\times\sigma) \\
\chi_d\,d\mu &= \chi_d w\,d\sigma_2 + \chi_d w_R\,d(\sigma\times\eta_2) \\
&\quad +\chi_d w_T\,d(\eta_1\times\sigma) + \chi_d\,d\lambda.
\end{aligned}
$$

Direct comparison of the above with the definitions of the component measures shows that the singular parts of all these measures are already consistent with property (10); that is, the assumption of (10) affects only the behavior of w. There are nine cases, depending on the values of $\sigma(A)$ and $\sigma(B)$:

case	$\sigma(A)$	$\sigma(A^c)$	$\sigma(B)$	$\sigma(B^c)$
(I)	>0	>0	>0	>0
(II)	>0	>0	1	0
(III)	1	0	>0	>0
(IV)	>0	>0	0	1
(V)	0	1	>0	>0
(VI)	1	0	1	0
(VII)	0	1	1	0
(VIII)	1	0	0	1
(IX)	0	1	0	1

In cases (I), (II) and (III), the space $\mathcal{L}_a$ fails to be "doubly regular" since the logarithmic integrability condition part of Lemma 2 fails. Hence condition (10) could not hold. Thus we have shown that (10) implies that one of (IV)–(IX) must hold, or equivalently

$$\begin{aligned} \sigma(A^c) &= 1 \\ &\text{or} \\ \sigma(B^c) &= 1 \\ &\text{or} \\ \sigma(A) &= \sigma(B) = 1. \end{aligned} \tag{11}$$

Note that (11) is a simple measure-theoretic criterion.

Continuing, assume (11). Then one of the cases (IV)–(IX) must hold. If not (VI), then

$$\begin{aligned} \chi_F &= \chi_a + \chi_c \\ &= 0 + \chi_c \\ &= \chi_F \chi_{E^c} \\ &\in \mathcal{M}(R_0), \end{aligned}$$

and similarly $\chi_E \in \mathcal{M}(T_0)$. On the other hand if (VI) holds, let Ω and Υ be measurable sets of $\mathbf{T}$ such that $\sigma(\Omega) = \sigma(\Upsilon) = \eta_1(\Omega^c) = \eta_2(\Upsilon^c) = 0$. Now

$$\begin{aligned} \chi_F &= \chi_a + \chi_c \\ &= \chi_a + \chi_b \perp \chi_b + \chi_c \\ &= \chi_E \perp \chi_E \chi_{(\mathbf{T}\times\Upsilon)} + \chi_{(E^c \cap F)} \\ &\in \mathcal{M}(R_0); \end{aligned}$$

in a similar way we get

$$\begin{aligned}\chi_E &= \chi_F \perp \chi_F \chi_{(\Omega\times\mathbf{T})} + \chi_{(E\cap F^c)}\\ &\in \mathcal{M}(T_0).\end{aligned}$$

We conclude that (11) implies (12):

$$\begin{aligned}\chi_E &\in \mathcal{M}(T_0)\\ \chi_F &\in \mathcal{M}(R_0).\end{aligned}\tag{12}$$

The condition (12) says that the set E associated with horizontal regularity is well behaved with respect to vertical dynamics, and vice-versa.

Next, let us write

$$\begin{aligned}\chi_a &= \chi_{E\cap F}\\ \chi_b &= \chi_{E\cap F^c}\\ \chi_c &= \chi_{E^c\cap F}\\ \chi_d &= \chi_{E^c\cap F^c}.\end{aligned}$$

It is always true that χ_c and χ_d belong to $\mathcal{S}_R$; likewise χ_b and χ_d belong to $\mathcal{S}_T$. If we assume that (12) holds, then all of χ_a, χ_b, χ_c and χ_d lie in $\mathcal{M}(R_0)\cap\mathcal{M}(T_0)$. Let $f\in\mathcal{M}(R_0)$. There are finite trigonometric sums $p_n \hat{\in} R_0$ such that

$$\int |f \perp p_n|^p \, d\mu \to 0.$$

Consequently,

$$\begin{aligned}\int |f \perp p_n|^p \chi_a \, d\mu &\to 0\\ \int |\chi_a f \perp \chi_a p_n|^p \, d\mu &\to 0.\end{aligned}$$

Since $\chi_a p_n$ belongs to $\mathcal{M}(R_0)$, this shows that $\chi_a f$ does as well. In conclusion, $\chi_a\mathcal{M}(R_0)$ is a subspace of $\mathcal{M}(R_0)$. Similar statements hold with indices b, c and d, and with R_0 replaced by T_0. With the obvious shorthand, we may express this as follows.

$$\begin{aligned}\mathcal{M}(R_0) &= \mathcal{M}_a(R_0)\oplus_p\mathcal{M}_b(R_0)\oplus_p\mathcal{M}_c(R_0)\oplus_p\mathcal{M}_d(R_0)\\ \mathcal{M}(T_0) &= \mathcal{M}_a(T_0)\oplus_p\mathcal{M}_b(T_0)\oplus_p\mathcal{M}_c(T_0)\oplus_p\mathcal{M}_d(T_0).\end{aligned}$$

Now for any $f \in L^p(\mu)$ consider

$$\begin{aligned}
& \inf\{\int |f+g|^p \, d\mu : \quad g \mathrel{\hat{\in}} R\} \\
= \; & \inf\{\int |\chi_a f + (1 \perp \chi_a) f + \chi_a g + (1 \perp \chi_a) g|^p \, d\mu : \quad g \mathrel{\hat{\in}} R\} \\
= \; & \inf\{\int |\chi_a f + \chi_a g_1|^p \, d\mu : \quad g_1 \mathrel{\hat{\in}} R\} \\
& + \inf\{\int |(1 \perp \chi_a) f + (1 \perp \chi_a) g_2|^p \, d\mu : \quad g_2 \mathrel{\hat{\in}} R\}.
\end{aligned}$$

Indeed equality holds in the last step because $\chi_a g_1$ and $(1 \perp \chi_a) g_2$ can be varied independently within $\mathcal{M}(R)$. This shows that if f lies in $\mathcal{L}_a$, then its metric projection into $\mathcal{M}(R)$ (in the geometry of $L^p(\mu)$) already belongs to $\mathcal{M}_a(R)$. The same is true with the other indices b, c and d, and with R replaced by T. We can express these statements as follows. Let $P(R)$ be the metric projection from $L^p(\mu)$ into $\mathcal{M}(R)$, and $P_a(R)$ the corresponding metric projection in $\mathcal{L}_a$; likewise with indices b, c and d. Furthermore we write $P|\mathcal{L}$ for the restriction of P to $\mathcal{L}$.

We have shown that (12) implies the condition

$$\left.\begin{array}{ll}
P(R)|\mathcal{L}_a = P_a(R) & P(T)|\mathcal{L}_a = P_a(T) \\
P(R)|\mathcal{L}_b = P_b(R) & P(T)|\mathcal{L}_b = P_b(T) \\
P(R)|\mathcal{L}_c = P_c(R) & P(T)|\mathcal{L}_c = P_c(T) \\
P(R)|\mathcal{L}_d = P_d(R) & P(T)|\mathcal{L}_d = P_d(T).
\end{array}\right\} \tag{13}$$

In essence, (13) says that the taking of component spaces respects both vertical and horizontal halfplane projections.

Finally, let $P(\mathcal{S}_R)$ and $P(\mathcal{S}_T)$ be the metric projections onto $\mathcal{S}_R$ and $\mathcal{S}_T$, respectively. Let $P_a(\cdot)$ be the metric projection of $L^p(\chi_a \, d\mu)$ onto the space $(\cdot)$, and define similarly with the other indices. From (13) we easily deduce

$$\begin{array}{ll}
P(\mathcal{S}_R)|\mathcal{L}_a = P_a(\mathcal{S}_R) & P(\mathcal{S}_T)|\mathcal{L}_a = P_a(\mathcal{S}_T) \\
P(\mathcal{S}_R)|\mathcal{L}_b = P_b(\mathcal{S}_R) & P(\mathcal{S}_T)|\mathcal{L}_b = P_b(\mathcal{S}_T) \\
P(\mathcal{S}_R)|\mathcal{L}_c = P_c(\mathcal{S}_R) & P(\mathcal{S}_T)|\mathcal{L}_c = P_c(\mathcal{S}_T) \\
P(\mathcal{S}_R)|\mathcal{L}_d = P_d(\mathcal{S}_R) & P(\mathcal{S}_T)|\mathcal{L}_d = P_d(\mathcal{S}_T).
\end{array}$$

We then get

$$\begin{aligned}
\mathcal{L}_{ab} & \subseteq P_a(\mathcal{S}_R)\mathcal{L}_a = P(\mathcal{S}_R)\mathcal{L}_a = (0) \\
\mathcal{L}_{ac} & \subseteq P_a(\mathcal{S}_T)\mathcal{L}_a = P(\mathcal{S}_T)\mathcal{L}_a = (0) \\
\mathcal{L}_{ad} & \subseteq P_a(\mathcal{S}_R)\mathcal{L}_a = P(\mathcal{S}_R)\mathcal{L}_a = (0),
\end{aligned}$$

and so $\mathcal{L}_{aa} = \mathcal{L}_a$. Corresponding statements are true with the other indices. This proves that (13) implies (10). This circle of implications yields the theorem. □

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